Journal of Engineering Physics and Thermophysics, Vol. 81, No. 3, 2008

ASYMPTOTICS OF EIGENVALUES FOR THE LAPLACE EQUATION IN DOMAINS WITH SLOTS

I. M. Martynenko UDC 517.9:539.3

The asymptotic formula for the eigenvalues of the Laplace equation in three-dimensional domains bounded by a finite number of closed and open nonintersecting Lyapunov surfaces with smooth boundaries has been derived. This result is of importance in calculating stationary thermal conductivity in solid bodies in the presence of slots in them.

The eigenvalues of the Dirichlet problem for the Laplace equation have adequately been investigated in domains bounded by smooth closed Lyapunov surfaces using the classical Green functions [1–3]. Below, we consider D domains whose boundaries are the finite number of closed and open nonintersecting Lyapunov surfaces $S = \Sigma \cup \sigma$, where Σ is the finite number of closed surfaces, $\Sigma = \bigcup\limits_k \Sigma_k$, $k \in \overline{0, K}$, and $\sigma = \bigcup\limits_n \sigma_n$, $n = \overline{1, N}$. It is assumed that σ_n are the two-sided surfaces bounded by smooth curves Γ_n, and Σ_k and σ_n are contained within Σ_0 for $n \in \overline{1, N}$ and $k \in \overline{1, K}$. We will call σ_n slots. For solution of the Dirichlet and Neumann problems in domains with slots, two-valued potentials with branch lines Γ_n have been introduced in [4, 5]; the jump formulas has been derived and the solvability of the fundamental boundary-value problems for a space with a cut (slot) nas been proved using them. It follows that the classical Green function $G_n^*(P, Q)$ exists for a space with a slot σ_n; this function will be written in the following manner: it is completely defined and continuous in the domain $E_3 \setminus \sigma_n$ when $P \neq Q$ and, as a function of the point P, is the solution of the Laplace equation vanishing for $P \in \sigma_n$; when $P = Q$ it has a point singularity of the form $\frac{1}{4\pi R}$, $R = |\overline{PQ}|$ is the distance between the points P and Q. Therefore, $G_n^*(P, Q)$ beyond σ_n is representable in the form

$$G_n^*(P, Q) = \frac{1}{4\pi R} + g_n(P, Q) = \omega_n(P, Q) + g_n'(P, Q) . \tag{1}$$

Formula (1) yields, for determining $g_n(P, Q)$ and $g_n'(P, Q)$, the equalities

$$g_n(P, Q) = -\frac{1}{4\pi R}, \quad g_n'(P, Q) = -\omega_n(P, Q), \quad P \in \sigma_n , \tag{2}$$

where $\omega_n(P, Q)$ is the Green function of a two-sheeted Riemannian space with a smooth branch line Γ_n [4]. Such functions were introduced by Sommerfeld [6] using the following conditions: $\omega(P, Q)$ as a function of the point P is completely defined and continuous in the Riemannian space except for the point Q at which it has a singularity of the form $\frac{1}{4\pi R}$, R is the distance between the points P and Q, $R = |PQ|$, is regular at infinity, and completely satisfies the Laplace equation in the Riemannian space except for the point Q and the branch line. Since σ_n is a two-sided open surface, relations (2) are possible only when $g_n(P, Q)$ and $g_n'(P, Q)$ are the two-valued functions with branch lines Γ_n. This yields the following representation of the harmonic function $V_n(P)$ in $E_3 \setminus \sigma_n$:

Belarusian State University, 4 Nezavisimost' Ave., Minsk, 220050, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 81, No. 3, pp. 577–582, May–June, 2008. Original article submitted September 4, 2006.

$$V_n(P)=\iint\limits_{\sigma_n} V_n(Q)\frac{\partial G_n^*(P,Q)}{\partial \mathbf{n}}ds=\iint\limits_{\sigma_n} V_{n+}(Q)\frac{\partial G_n^*(P,Q_+)}{\partial \mathbf{n}}ds_Q-\iint\limits_{\sigma_n} V_{n-}(Q)\frac{\partial G_n^*(P,Q_-)}{\partial \mathbf{n}}ds_Q\,, \tag{3}$$

where $\mathbf{n}$ is a fixed normal to σ_n; $V_{n+}(Q)$ and $V_{n-}(Q)$ represent the limiting $V_n(Q)$ values with the approximation $Q\to\sigma_n$ "from above" and "from below," i.e., in the direction of the positive and negative normals $\mathbf{n}$.

For the multiply connected domain D, we have the following formula:

$$V(P)=\sum_{k=0}^{K}\iint\limits_{\Sigma_k} u_k(Q)\frac{\partial G_k(P,Q)}{\partial n_Q}ds_Q+\sum_{i=1}^{N}\iint\limits_{\sigma_i} u_i(Q)\frac{\partial G_i^*(P,Q)}{\partial n_Q}ds_Q\,. \tag{4}$$

Here the integral with respect to σ_i is understood in the same way as that in formula (3), G_k is the classical Green function for the domain $D_k = E_3\setminus\Sigma_k$, G_i^* is the Green function for the domain $E_3\setminus\sigma_i$, and u_k and u_i are the unknown densities to determine which we should use the boundary conditions

$$V(P)_{symbol\Sigma_k}=f_k(P)\,,\quad P\in\Sigma_k\,;\quad V(P)_{symbol_{\sigma_i^\pm}}=\varphi_i^\pm(P)\,,\quad P\in\sigma_i\,. \tag{5}$$

Expressions (4)–(5) yield the following system of integral Fredholm equations for determination of these densities:

$$u_l(P)+\sum_{k\neq l=0}^{K}\iint\limits_{\Sigma_k} u_k(Q)\frac{\partial G_k(P,Q)}{\partial n_Q}ds_Q+\sum_{i=1}^{N}\iint\limits_{\sigma_i} u_i(Q)\frac{\partial G_i^*(P,Q)}{\partial n_Q}ds_Q=f_k(P)\,,$$
$$u_j(P)+\sum_{k=0}^{K}\iint\limits_{\Sigma_k} u_k(Q)\frac{\partial G_k(P,Q)}{\partial n_Q}ds_Q+\sum_{i\neq j=1}^{N}\iint\limits_{\sigma_i} u_i(Q)\frac{\partial G_i^*(P,Q)}{\partial n_Q}ds_Q=\varphi_j(P)\,. \tag{6}$$

The following theorem holds: *every continuous harmonic function bounded in domain D and whose first derivatives in the vicinity of the branch line* Γ_i *act as* $O(1/R_i^\alpha)$, $0\le\alpha<1$ *and* R_i *is the distance from P to* Γ_i, *is representable uniquely in the form of the sum of harmonic functions in singly connected domains* D_0, D_1, ..., D_n, T_1, ..., T_N, $T_k = E_3\setminus\sigma_k$, $k = \overline{1,N}$. *In unbounded domains* D_1, ..., D_n *and* T_1, ..., T_N, *the harmonic functions sought are regular at infinity, i.e., uniformly tend to* 0 *when* $|x|\to\infty$.

First we note that the Green formula for domain D yields the following representation:

$$V(x)=\sum_{k=0}^{n}V_k(x)+\sum_{i=1}^{N}W_i(x)\,, \tag{7}$$

where

$$V_k(x)=\iint\limits_{\Sigma_k}\left(\omega(x,y)\frac{\partial V}{\partial n_y}-V(y)\frac{\partial\omega(x,y)}{\partial n_y}\right)d_yS\,;\quad W_i(x)=\iint\limits_{\sigma_i}\left(\omega(x,y)\frac{\partial W}{\partial n_y}-W(y)\frac{\partial\omega(x,y)}{\partial n_y}\right)d_yS\,. \tag{8}$$

Here $\omega(x, y)$ is the Green function of the Riemannian space whose branch lines are $\Gamma = \cup_{i=1}^N\Gamma_i$. In the integral with respect to Σ_k, we have a term of the regular part of the function $\omega(x, y)$, which is identically equal to zero. The uniqueness of the representation (7) is easily proved by reduction to absurdity. Indeed, assuming the existence of these two representations in one harmonic function $V(x)$, we obtain, for their difference, the following equality:

$$\sum_{k=0}^{n}\overline{V}_k(x)+\sum_{i=1}^{N}\overline{W}_i(x)=0\,,\quad \overline{V}_k(x)=V_k^1(x)-V_k^2(x)\,,\quad \overline{W}_i(x)=W_i^1(x)-W_k^2(x)\,, \tag{9}$$

whence

$$\overline{V}_0(x) = -\sum_{k=1}^{n} \overline{V}_k(x) - \sum_{i=1}^{N} \overline{W}_i(x) . \tag{10}$$

It follows from this formula that $\overline{V}_0(x)$ is a regular harmonic function throughout the Euclidean space E_3; therefore, it is identically equal to zero. Quite analogously we prove that $V_k(x) \equiv 0$ and $W_i(x) \equiv 0$, $k = \overline{1, n}$ and $i = \overline{1, N}$. The existence of the solution of the initial Dirichlet problem follows from the Fredholm character of the system of equations (6).

We denote the Green function of the Dirichlet problem for the Laplace equation in domain D with boundary $S = \Sigma \cup \sigma$ (its existence follows from the existence of the solution of the Dirichlet problem for the Laplace equation in domains with slots) by $G(P, Q)$. We assume that $\lambda = \text{const} > 0$. Then the problem

$$\Delta v - \lambda v = -\varphi(P), \quad P \in D, \quad v\big|_S = 0 \tag{11}$$

is equivalent to the following integral equation [2]:

$$v(P) = -\lambda \iiint\limits_D G(P, Q)\, v(Q)\, d_Q\tau + \iiint\limits_D G(P, Q)\, \varphi(Q)\, d_Q\tau . \tag{12}$$

The integral equation (12) has a unique solution for the function $\varphi(P)$ continuously differentiable in domain D and continuous up to boundary D.

We denote the Green function of the Dirichlet problem for the equation $\Delta v - \lambda v = 0$ in domain D by $G_1(P, Q, \lambda)$. Then we obtain

$$G_1(P, Q, \lambda) = \frac{\exp(-\sqrt{\lambda}\, r)}{4\pi r} + g_1(P, Q, \lambda), \quad g_1(P, Q, \lambda)_{symbols_S} = -\left.\frac{\exp(-\sqrt{\lambda}\, r)}{4\pi r}\right|_S .$$

The solution of Eq. (11), which vanishes in 0 on S, can be represented in the following form:

$$v(P) = \iiint\limits_D G_1(P, Q, \lambda)\, \varphi(Q)\, d_Q\tau .$$

By the ordinary method, it is proved that $G_1(P, Q, \lambda) = G_1(Q, P, \lambda)$, $G(P, Q) = G(Q, P)$, $0 < G_1(P, Q, \lambda) < \exp(-\sqrt{\lambda}\, r)/(4\pi r)$, $r = |PQ|$, and $P \in D$.

It follows from what has been stated above that the Green function $G_1(P, Q, \lambda)$ is the resolvent of the integral equation (2). This leads to the following equality:

$$G_1(P, Q, \lambda) = G(P, Q) - \lambda \iiint\limits_D G(P, Q')\, G_1(Q', Q, \lambda)\, d_{Q'}\tau . \tag{13}$$

Therefore, from the representation of the resolvent for a symmetric integral equation we have [2]

$$G_1(P, Q, \lambda) = G(P, Q) - \lambda \sum_{k=1}^{\infty} \frac{v_k(P)\, v_k(Q)}{\lambda_k(\lambda_k + \lambda)} . \tag{14}$$

Here λ_k and $v_k(P)$ are the eigenvalues and eigenfunctions of the kernel $G(Q, P)$, i.e., the equations $\Delta v - \lambda v = 0$ in domain D with a boundary condition v/S. Therefore, we obtain

$$\iiint\limits_D G(P, Q') \, G_1(Q', Q, \lambda) \, d_{Q'}\tau = \sum_{k=1}^{\infty} \frac{v_k(P) \, v_k(Q)}{\lambda_k(\lambda_k + \lambda)} \, . \tag{15}$$

We denote the coefficients of the Fourier series of the function $G_1(Q', Q, \lambda)$ in the functions $v_k(Q')$ by h_k:

$$h_k = \iiint\limits_D v_k(Q') G_1(Q', Q, \lambda) \, d_{Q'}\tau \, .$$

Since $v_k(Q')$ is the solution of the equation $\Delta v_k - \lambda_k v_k = 0$, we obtain

$$\lambda_k h_k = \iiint\limits_D \Delta v_k(Q') G_1(Q', Q, \lambda) \, d_{Q'}\tau \, ,$$

$$(\lambda_k + \lambda) \, h_k = \iiint\limits_D G_1(Q', Q, \lambda) \; (\Delta v_k(Q') - \lambda v_k(Q')) \, d_{Q'}\tau \, .$$

The last formula yields [2]

$$h_k = \frac{v_k(Q)}{\lambda_k + \lambda} \, , \quad \sum_{k=1}^{\infty} \frac{v_k^2(P)}{\lambda_k^2} = \iiint\limits_D G^2(P, Q) \, d_Q\tau \, ,$$

$$\sum_{k=1}^{\infty} \frac{v_k^2(P)}{\lambda_k(\lambda_k + \lambda)} = \iiint\limits_D G(P, Q') \, G_1(Q', P, \lambda) \, d_{Q'}\tau \, .$$

Integrating the last equality with respect to D, we arrive at the following (since $\iiint\limits_D v_k^2(P) d_P\tau = 1$):

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k(\lambda_k + \lambda)} = \iiint\limits_D \psi(P, \lambda) \, d\tau \, , \tag{16}$$

where

$$\psi(P, \lambda) = \iiint\limits_D G(P, Q) \, G_1(Q, P, \lambda) \, d_Q\tau \, . \tag{17}$$

Since

$$G_1(Q, P, \lambda) = \frac{\exp(-\sqrt{\lambda}\, r)}{4\pi r} + g_1(Q, P, \lambda) \, , \quad G_1(Q, P, \lambda)\big|_S = 0 \, , \quad r = |PQ| \, ,$$

we have, within D, the estimates

$$0 < G_1(Q, P, \lambda) < \frac{\exp(-\sqrt{\lambda}\, r)}{4\pi r} \, , \quad 0 \geq g_1(P, Q, \lambda) \geq -\frac{\exp(-\sqrt{\lambda}\, r)}{4\pi r} \, . \tag{18}$$

Analogous estimates are true of $G(P, Q)$ and $g(P, Q)$. It follows from them that

$$\mathit{symbol}\,\psi(P, \lambda)\,\mathit{symbol} \leq \iiint\limits_{D_i} \frac{\exp(-\sqrt{\lambda}\, r)}{16\pi^2 r^2} \, d\tau_Q \leq \frac{1}{16\pi^2} \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \exp(-\sqrt{\lambda}\, r) \sin\theta \, dr d\theta d\varphi = \frac{1}{4\pi\sqrt{\lambda}} \, .$$

Furthermore, $\sqrt{\lambda}\,\psi(P,\lambda) \to \dfrac{1}{4\pi}$ is uniform in every closed subdomain $D' \in D$. Therefore, we obtain

$$\lim_{\lambda\to+\infty} \iiint_D \sqrt{\lambda}\,\psi(P,\lambda) = \frac{v}{4\pi}, \quad \lim_{\lambda\to+\infty} \sqrt{\lambda} \sum_{k=1}^{\infty} \frac{1}{\lambda_k(\lambda_k+\lambda)} = \frac{v}{4\pi}. \tag{19}$$

We use the following theorem [2]:

T h e o r e m. *If the series*

$$s(\lambda) = \sum_{k=1}^{\infty} \frac{c_k}{\lambda_k+\lambda}, \quad \text{where } c_k > 0, \ \lambda_k > 0, \ 0 \le \lambda_1 \le \lambda_2 \le \ldots, \ \lambda_n \to \infty$$

converges for $\lambda > 0$ *and*

$$\lim_{\lambda\to+\infty} \sqrt{\lambda}\, s(\lambda) = H,$$

$$\lim_{\lambda\to+\infty} \frac{1}{\sqrt{\lambda}} \sum_{\lambda_k\le\lambda} c_k = \frac{2H}{\pi}, \tag{20}$$

in the last sum, summation extends to those k values for which $\lambda_k \le \lambda$.

As applied to the series (19), we set $c_k = \dfrac{1}{\lambda_k}$ and $H = \dfrac{v}{4\pi}$ and from equality (20) we have

$$\lim_{\lambda\to+\infty} \frac{1}{\sqrt{\lambda}} \sum_{\lambda_k\le\lambda} \frac{1}{\lambda_k} = \frac{v}{2\pi^2}, \tag{21}$$

whence

$$\sum_{\lambda_k\le\lambda} \frac{1}{\lambda_k} = \frac{v}{2\pi^2}\sqrt{\lambda} + \varepsilon(\lambda)\sqrt{\lambda},$$

where $\varepsilon \to 0$ for $\lambda \to \infty$ or for $\lambda = \lambda_n$,

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} = \frac{v}{2\pi^2}\sqrt{\lambda_n} + \varepsilon_n\sqrt{\lambda_n}.$$

Introducing the notation

$$\sigma_n = \sum_{k=1}^{n} \frac{1}{\lambda_k},$$

we write

$$n = \sum_{k=1}^{n} \lambda_k \frac{1}{\lambda_k} = \sigma_1(\lambda_1-\lambda_2) + \sigma_2(\lambda_2-\lambda_3) + \ldots + \sigma_{n-1}(\lambda_{n-1}-\lambda_n) + \sigma_n\lambda_n$$

and for the nondecreasing function

$$\varphi(\lambda) = \frac{v}{2\pi^2}\sqrt{\lambda} + \varepsilon(\lambda)\sqrt{\lambda},$$

we obtain

$$\int_0^{\lambda_n} \varphi(\lambda)\,d\lambda = \sigma_1(\lambda_1 - \lambda_2) + \sigma_2(\lambda_2 - \lambda_3) + \ldots + \sigma_{n-1}(\lambda_{n-1} - \lambda_n) = \frac{v}{3\pi^2}\lambda_n^{3/2} + \int_0^{\lambda_n} \varepsilon(\lambda)\sqrt{\lambda}\,d\lambda.$$

From the determination of $\varepsilon(\lambda)$, we have

$$\frac{1}{\lambda_n^{3/2}}\int_0^{\lambda_n} \varepsilon(\lambda)\sqrt{\lambda}\,d\lambda \to 0 \quad \text{at} \quad n \to +\infty.$$

Therefore, we obtain the following expression for n:

$$n = \frac{v}{6\pi^2}\lambda_n^{3/2} + \varepsilon_n\lambda_n^{3/2},$$

whence

$$\lambda_n = \left(\frac{6\pi^2 n}{v}\right)^{3/2} + \varepsilon_n n^{3/2}, \tag{22}$$

where $\varepsilon_n \to 0$ for $n \to \infty$, v is the volume of domain D whose boundary contains smooth closed surfaces S_k, $k = \overline{0, N}$ and smooth open surfaces (slots σ_m, $m = \overline{1, M}$) bounded by smooth curves Γ_m.

Formula (22) was obtained earlier by Weyl [1] for domains bounded by smooth closed surfaces Σ, and it has been established for the first time in the case of domains with slots.

NOTATION

P, point E_3, x, its radius vector; Q, point E_3, y, its radius vector; S, surface; s, S-surface element; v, volume of domain D; λ_n, eigenvalues of the Dirichlet problem.

REFERENCES

1. H. Weyl, *Selected Papers. Mathematics*, *Theoretical Physics* [Russian translation], Nauka, Moscow (1984).
2. V. I. Smirnov, *A Course in Higher Mathematics* [in Russian], Vol. 4, Pt. 2, Nauka, Moscow (1981).
3. M. Sh. Birman and M. Z. Solomyak, The asymptotics of the spectrum of differential equations, in: *Progress in Science and Technology. "Mathematical Analysis"* [in Russian], Vol. 14, Nauka, Moscow (1977).
4. M. D. Martynenko, Basic boundary-value problems of the theory of potential for slotted regions, *Ukr. Mat. Zh.*, **15**, No. 4 (1963).
5. M. A. Zhuravkov and M. D. Martynenko, *Singular Solutions and Integral Equations in the Mechanics of Deformable Media* [in Russian], BGU, Minsk (1999).
6. A. Sommerfeld, Über verzweigte Potential in Raum, *Proc. London Math. Soc.*, **28** (1887).